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QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

34. [*1917*, 134, 341; *1920*, 114, 301.] Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

REPLY BY D. C. GILLESPIE, Cornell University.

The formula is a case of Simpson's Rule, where only two intervals are used. If $f(x)$ is a polynomial of degree not greater than three, this approximating formula overplays its rôle and evaluates the integral exactly. The function $f(x)$ being assumed continuous, one might surmise that the formula would not evaluate the integral exactly, for all values of h, unless $f(x)$ were a polynomial of degree three or less. The correctness of this surmise has not been demonstrated. In this note it is shown that *if* $f(x)$ *is continuous,* $0 \leqq x \leqq b$, *and has six continuous derivatives,* $0 \leqq x \leqq H < b$, *and the formula evaluates the integral for each value of* h, $0 \leqq h \leqq b$, *then* $f(x)$ *is a polynomial of degree not higher than three,* $0 \leqq x \leqq b$.

Suppose first that a function $g(x)$, continuous in $(0, b)$, satisfies the functional equation

$$\int_0^h x^n g(x)dx = \frac{h}{6}\left\{4\left(\frac{h}{2}\right)^n g\left(\frac{h}{2}\right) + h^n g(h)\right\} \tag{1}$$

for $0 \leqq h \leqq b$ and n a positive integer. The mean value theorem yields

$$\frac{h^{n+1}}{n+1} g(\theta h) = h^{n+1}\left\{\frac{1}{3}\cdot\frac{1}{2^{n-1}} g\left(\frac{h}{2}\right) + \frac{g(h)}{6}\right\}, \qquad 0 < \theta < 1. \tag{2}$$

The function $g(x)$ is continuous; hence, on dividing through by h^{n+1} and taking the limit of both sides as h approaches zero, we obtain

$$\frac{g(0)}{n+1} = \frac{1}{3}\cdot\frac{1}{2^{n-1}} g(0) + \frac{1}{6} g(0).$$

If $g(0) \neq 0$ we have

$$\frac{1}{n+1} = \frac{1}{3}\cdot\frac{1}{2^{n-1}} + \frac{1}{6}.$$

This equation is satisfied for n equal to 1 or 2 or 3 and for no other positive integral value of n. Hence, if n is to be greater than three, $g(0)$ must be zero.

We assume next $g(0) = 0$ and again find an upper limit for n. Let $\bar{h}$ be a value of x for which $|g(x)|$ assumes its upper limit in $(0, b)$. Equation (2) may now be written

$$\frac{\bar{h}^{n+1}}{n+1} g(\theta\bar{h}) = \bar{h}^{n+1}\left\{\frac{1}{3}\cdot\frac{1}{2^{n-1}} g\left(\frac{\bar{h}}{2}\right) + \frac{g(\bar{h})}{6}\right\}, \qquad 0 < \theta < 1. \tag{2'}$$

or

$$\frac{g(\theta\bar{h})}{n+1} = \frac{1}{3}\cdot\frac{1}{2^{n-1}} g\left(\frac{\bar{h}}{2}\right) + \frac{g(\bar{h})}{6}.$$

But

$$\left|g\left(\frac{\bar{h}}{2}\right)\right| \leqq |g(\bar{h})| \quad \text{and} \quad |g(\theta\bar{h})| \leqq |g(\bar{h})|$$

hence

$$\left|\frac{g(\bar{h})}{n+1}\right| \geqq \left|\frac{1}{3}\cdot\frac{1}{2^{n-1}} g\left(\frac{\bar{h}}{2}\right) + \frac{g(\bar{h})}{6}\right| \geqq \left(\frac{1}{6} - \frac{1}{3}\cdot\frac{1}{2^{n-1}}\right)|g(\bar{h})| \text{ for } n \geqq 2.$$

Now

$$\frac{1}{n+1} < \frac{1}{6} - \frac{1}{3} \cdot \frac{1}{2^{n-1}} \text{ for } n \geqq 6.$$

Equation (2′) then can not be satisfied for $n \geqq 6$, unless, of course, $g(x)$ is identically zero.

Suppose now that a function $f(x)$ which satisfies the given equation

$$(3) \qquad \int_0^h f(x)dx = \frac{h}{6}\left\{ f(0) + 4f\left(\frac{h}{2}\right) + f(h) \right\}, \qquad 0 \leqq h \leqq b$$

is continuous in $(0, b)$ and has six continuous derivatives in an interval $(0, H)$ in $(0, b)$. The Taylor expansion theorem applied for values of x in $(0, H)$ gives

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{\text{iv}}(\theta x)\frac{x^4}{4!}, \qquad 0 < \theta < 1.$$

The quantity θ is, to be sure, not a constant but a function of x; nevertheless $f^{\text{iv}}(\theta x)$ is a continuous function of x in $(0, H)$. Susbtituting in equation (3) for $f(x)$ this expansion we obtain

$$\int_0^h \frac{x^4}{4!} f^{\text{iv}}(\theta x)dx = \frac{h}{6}\left\{ 4\left(\frac{h}{2}\right)^4 \frac{f(\theta h/2)}{4!} + \frac{h^4 f(\theta h)}{4!} \right\}.$$

This equation, as we have already seen, requires that $f^{\text{iv}}(0) = 0$; and one shows in the same way that $f^{\text{v}}(0) = 0$. Taking the sixth degree term in the expansion,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{\text{vi}}(\theta x)\frac{x^6}{6!}, \qquad 0 < \theta < 1.$$

If this expansion for $f(x)$ is substituted in (3) there results

$$\int_0^h \frac{x^6 f^{\text{vi}}(\theta x)}{6!} dx = \frac{h}{6}\left\{ 4\frac{h^6}{2^6}\frac{f^{\text{vi}}(\theta h/2)}{6!} + \frac{h^6 f^{\text{vi}}(\theta h)}{6!} \right\}.$$

We have shown that this equation can hold only when $f^{\text{vi}}(\theta x)$ is zero for all values of x in $(0, H)$. In the interval $(0, H)$ then $f(x)$ coincides with a cubic polynomial.

In Questions and Discussions *1920*, 302, it was pointed out by the Editor that two continuous functions satisfying equation (3) and coinciding in some interval from the origin to the right coincide throughout. Thus the result announced at the beginning is established.

The theorem, just quoted and used in the proof, is capable of a slight extension; *i.e.*, if a continuous function $f(x)$ satisfies equation (3) and is zero for $a \leqq x \leqq c$ where $c > 2a > 0$, then it is zero throughout $(0, b)$. For suppose $c \geqq h \geqq 2a$, then

$$\int_0^h f(x)dx = \frac{h}{6}f(0).$$

Since, however, $f(x) = 0$, $a \leqq x \leqq c$, as h increases from $2a$ to c the integral remains constant, hence $f(0) = 0$. Thus for $a \leqq h \leqq c$,

$$\int_0^h f(x)dx = \frac{2}{3}hf\left(\frac{h}{2}\right),$$

and as the integral maintains from $h = a$ to $h = c$ a constant value, which we have seen must be zero,

$$f\left(\frac{h}{2}\right) = 0, \qquad a \leqq h \leqq c,$$

or

$$f(x) = 0, \qquad \frac{a}{2} \leqq x \leqq \frac{c}{2}.$$

Continuing this process, we see that $f(x) = 0$, $0 \leqq x \leqq c$. Then by the previously quoted remark of the Editor, $f(x) = 0$ in the entire interval $(0, b)$.